Characterization of Product Measures by Integrability Condition

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1 Introduction

Let $(\mathcal{E}', \mu)$ be the real Gaussian space, where $\mathcal{E}'$ is the space of tempered distributions and μ be the standard Gaussian measure on $\mathcal{E}'$. In the recent papers [4, 7, 8], Asai, Kubo and Kuo (AKK for short) have shown that in order to construct the Gel'fand triple $[\mathcal{E}]_u \subset L^2(\mathcal{E}', \mu) \subset [\mathcal{E}]_u^*$ associated with a growth function $u \in C_{+,1/2}$, essential conditions on u are (U0)(U2)(U3) stated in Section 2.2. Legendre transform and dual Legendre transform (Section 2.1) play important roles to get this result. We note that Gannoun et al. [11] have obtained similar results independently. Some relationships with [11] are discussed in Section 2.2. In addition, the intrinsic topology for $[\mathcal{E}]_u$ has been given and the characterization theorem for positive Radon measures on $\mathcal{E}'$ has also been proved by considering an integrability condition [5, 8].

Now it is natural to ask whether "positivity" of white noise operators can be discussed in some sense and characterized. To answer this question, we consider the Gel'fand triple over the Complex Gaussian space $(\mathcal{E}'_c, \mu_c)$, i.e. $\mathcal{E}'_c = \mathcal{E}' + i\mathcal{E}'$ equipped with the product measure $\mu_c = \mu' \times \mu'$ where μ' is the Gaussian measure on $\mathcal{E}'$ with variance $1/2$ (Section 2.2). Following AKK's Legendre transform technique, we have $\mathcal{W}_{u_1,u_2} \subset L^2(\mathcal{E}'_c, \mu_c) \subset [\mathcal{W}]^*_{u_1,u_2}$ for functions $u_1, u_2 \in C_{+,1/2}$ satisfying (U0)(U2)(U3). Several examples for u_1, u_2 are given in Section 2.3. We remark that Ouerdiane [28] studied a special case $u_1(r^2) = u_2(r^2) = \exp(k^{-1}r^k)$, where $1 \leq k \leq 2$. In Section 3, the characterization theorem for measures can be extended to the case of positive product Radon measures on $\mathcal{E}' \times \mathcal{E}'$. In addition, the notion of pseudo-positive operators is naturally introduced via kernel theorem and characterized by an integrability condition. Lemma 3.2 plays crucial roles in Section 3.

2 White Noise Functions

2.1 Legendre Transform and Dual Legendre Transform

In this section we introduce the Legendre transform and dual Legendre transform which will be used for the constructions of the Gel'fand triples over the real and complex Gaussian spaces.

First, let us define two kinds of convex functions. A positive function f on $[0, \infty)$ is called

(1) *(log, exp)-convex* if the function $\log f(e^x)$ is convex on $\mathbf{R}$;

(2) *(log, x^k)-convex* if the function $\log f(x^k)$ is convex on $[0, \infty)$. Here $k > 0$.

Let $C_{+,\log}$ denote the collection of all positive continuous functions u on $[0, \infty)$ satisfying the condition:

$$\lim_{r\to\infty} \frac{\log u(r)}{\log r} = \infty.$$

The *Legendre transform* ℓ_u of $u \in C_{+,\log}$ is defined to be the function

$$\ell_u(t) = \inf_{r>0} \frac{u(r)}{r^t}, \qquad t \in [0, \infty).$$

Let $C_{+,\,1/2}$ denote the collection of all positive continuous functions u on $[0, \infty)$ satisfying the condition:

$$\lim_{r\to\infty} \frac{\log u(r)}{\sqrt{r}} = \infty.$$

The *dual Legendre transform* u^* of $u \in C_{+,1/2}$ is defined to be the function

$$u^*(r) = \sup_{s\ge 0} \frac{e^{2\sqrt{rs}}}{u(s)}, \qquad r \in [0, \infty).$$

Note that $C_{+,1/2} \subset C_{+,\log}$. Assume that $u \in C_{+,\log}$ and $\lim_{n\to\infty} \ell_u(n)^{1/n} = 0$. We define the *L-function* $\mathcal{L}_u$ of u by

$$\mathcal{L}_u(r) = \sum_{n=0}^{\infty} \ell_u(n) r^n. \tag{2.1}$$

For discussions in the rest of the paper, we will need the following facts in [7, 8]. See also [1].

Fact 2.1. *(1) Let $u \in C_{+,\log}$ be (log, exp)-convex. Then its L-function $\mathcal{L}_u$ is also (log, exp)-convex and for any $a > 1$,*

$$\mathcal{L}_u(r) \le \frac{ea}{\log a} u(ar), \qquad \forall r \ge 0.$$

(2) Let $u \in C_{+,\log}$ be increasing and (log, x^k)-convex. Then there exists a constant C, independent of k, such that

$$u(r) \le C\mathcal{L}_u(2^k r), \qquad \forall r \ge 0. \tag{2.2}$$

(3) Let $u \in C_{+,\log}$ be increasing and (log, x^k)-convex. Then for any $a > 1$, we have

$$\mathcal{L}_u(r) \le \sqrt{\ell_u(0)\frac{ea}{\log a}}\, u\big(a2^{k+1}r\big)^{1/2}. \tag{2.3}$$

Fact 2.2. *If $u \in C_{+,1/2}$ is (log, x^2)-convex, then the Legendre transform ℓ_{u^*} of u^* is given by*

$$\ell_{u^*}(t) = \frac{e^{2t}}{\ell_u(t)t^{2t}}, \qquad t \in [0,\infty).$$

2.2 Complex Gaussian Space

Let us start with taking a special choice of a Gel'fand triple:

$$\mathcal{E} = \mathcal{S}(\mathbf{R}) \subset \mathcal{E}_0 = L^2(\mathbf{R}, dt) \subset \mathcal{E}^* = \mathcal{S}^*(\mathbf{R})$$

just for convenience where $\mathcal{S}$ is the Schwarz space of rapidly decreasing functions and $\mathcal{S}^*$ is the space of tempered distributions. Consult [14, 21, 22, 23, 26] for more general setting. Let A be a positive self-adjoint operator in $\mathcal{E}_0$. So there exists an orthonormal basis $\{e_j\}_{j=0}^{\infty} \subset \mathcal{E}$ for $\mathcal{E}_0$ satisfying $Ae_j = \lambda_j e_j$. $|\cdot|_0$ denotes the norm of $\mathcal{E}_0$. For each $p \geq 0$ we define $|f|_p = |A^p f|_0$ and let $\mathcal{E}_p = \{f \in \mathcal{E}_0;\ |f|_p < \infty,\ p \geq 0\}$. Note that $\mathcal{E}_p$ is the completion of $\mathcal{E}$ with respect to the norm $|\cdot|_p$. Moreover,

$$\rho = \|A^{-1}\|_{OP}, \quad \|i_{q,p}\|_{HS}^2 = \sum_{j=0}^{\infty} \lambda_j^{-(q-p)} < \infty$$

for any $q > p \geq 0$. Then the projective limit space $\mathcal{E}$ of $\mathcal{E}_p$ as $p \to \infty$ is a nuclear space and the dual space of $\mathcal{E}$ is nothing but the inductive limit space $\mathcal{E}'$. Hence we have the following continuous inclusions:

$$\mathcal{E} \subset \mathcal{E}_p \subset \mathcal{E}_0 \subset \mathcal{E}_p' \subset \mathcal{E}', \quad p \geq 0,$$

where the norm on $\mathcal{E}_p'$ is given by $|f|_{-p} = |A^{-p}f|_0$. Throughout this paper, we denote the complexification of a real space X by X_c. Let μ' be the Gaussian measure on $\mathcal{E}'$ with variance 1/2, namely, a probability measure on $\mathcal{E}'$ given by the characteristic function:

$$e^{-\frac{1}{4}|\xi|_0^2} = \int_{\mathcal{E}'} e^{i\langle x,\xi\rangle} \mu'(dx), \qquad \xi \in \mathcal{E}.$$

Due to the topological isomorphism $\mathcal{E}_c' \cong \mathcal{E}' \times \mathcal{E}'$, we can define a probability measure $\mu_c = \mu' \times \mu'$ on $\mathcal{E}_c'$. The probability space $(\mathcal{E}_c', \mu_c)$ is called the *complex Gaussian space*, see [13]. We denote by $L^2(\mathcal{E}_c', \mu_c)$ the space of μ_c-square integrable functions on $\mathcal{E}_c'$. We should note that $L^2(\mathcal{E}_c', \mu_c) \cong L^2(\mathcal{E}', \mu') \otimes L^2(\mathcal{E}', \mu')$.

Let μ be the Gaussian measure on $\mathcal{E}'$ with variance 1 and $(\mathcal{E}', \mu)$ be the *real Gaussian space*. The next Fact 2.3 has been obtained in [8] for the Gel'fand triple $[\mathcal{E}]_u \subset L^2(\mathcal{E}', \mu) \subset [\mathcal{E}]_u^*$ associated with a growth function u. This triple is refered to as *the CKS-space with a weight sequence* $\alpha_u(n) = \big(\ell_u(n)n!\big)^{-1}$. For more precise discussion, we will need the following conditions on $u \in C_{+,1/2}$:

(U0) $\inf_{r \geq 0} u(r) = 1$.

(U1) u is increasing and $u(0) = 1$.

(U2) $\lim_{r\to\infty} r^{-1} \log u(r) < \infty$.

(U3) u is $(\log, x^2)$-convex.

Then we have

Fact 2.3. *Suppose $u \in C_{+,1/2}$ satisfies conditions (U0) (U2) (U3). Then the CKS-space with a weight sequence $\alpha_u(n)$ can be constructed. Moreover, characterization theorems hold.*

Remark. (1) We refer the reader to the papers of Asai et al. [7, 8] for details. We cite papers [2, 3, 9, 16, 17, 18, 19, 24, 29] for characterizaion theorems on papticular cases.
(2) We should mention here that our formulation has some links with a recent work by Gannoun et al. [11]. So let us explain some of them. Essential relationships are

$$u(r) = e^{2\theta(\sqrt{r})}, \quad u^*(r) = e^{2\theta^*(\sqrt{r})}$$

where $\theta^*(s) = \sup_{t>0}\{st - \theta(t)\}$ is adopted in [11]. In the following table we give the correspondence between our U-conditions and θ-conditions.

	u	θ
$(U0)$	$\inf_{r\geq 0} u(r) = 1$	$\inf_{r\geq 0} \theta(r) = 0$
$(U1)$	u is increasing and $u(0) = 1$	θ is increasing and $\theta(0) = 0$
$(U2)$	$\lim_{r\to\infty} \dfrac{\log u(r)}{r} < \infty$	$\lim_{r\to\infty} \dfrac{\theta(r)}{r^2} < \infty$
$(U3)$	u is $(\log, x^2)$-convex	θ is convex

2.3 CKS-space over Complex Gaussian Space

Next let us consider the Gel'fand triple over the complex white noise space $(\mathcal{E}_c', \mu_c)$ for our purpose.

$L^2(\mathcal{E}_c', \mu_c)$-norm $\|\varphi\|$ of φ is given by

$$\|\varphi\|^2 = \sum_{l,m=0}^{\infty} l!m!|f_{l,m}|_0^2, \qquad f_{l,m} \in \mathcal{E}_{0,c}^{\widehat{\otimes}(l+m)}.$$

In order to define norms in the spaces of test and generalized functions, we need a notation. For $\kappa_{l,m} \in (\mathcal{E}_c^{\otimes(l+m)})^*_{symm}$, we put

$$|\kappa_{l,m}|_{p_1,p_2} = |(A^{p_1})^{\otimes l} \otimes (A^{p_2})^{\otimes m} \kappa_{l,m}|_0.$$

For $p_1, p_2 \geq 0$ and given functions $u_1, u_2 \in C_{+,1/2}$ satisfying conditions $(U0)(U2)(U3)$, define the norm by

$$\|\varphi\|_{p_1,p_2}^2 = \sum_{l,m=0}^{\infty} \frac{1}{\ell_{u_1}(l)\ell_{u_2}(m)} |f_{l,m}|_{p_1,p_2}^2, \quad f_{l,m} \in \mathcal{E}_c^{\widehat{\otimes}(l+m)}. \tag{2.4}$$

Let $[\mathcal{E}_{p_1}]_{u_1} \otimes [\mathcal{E}_{p_2}]_{u_2} = \{\varphi \in L^2(\mathcal{E}_c', \mu_c) : \|\varphi\|_{p_1,p_2} < \infty\}$. Define the space $[\mathcal{E}]_{u_1} \otimes [\mathcal{E}]_{u_2}$ on $\mathcal{E}_c'$ to be the projective limit of $[\mathcal{E}_{p_1}]_{u_1} \otimes [\mathcal{E}_{p_2}]_{u_2}$ as $p_1, p_2 \to \infty$. For abbreviation, we put $\mathcal{W}_{u_1,u_2} = [\mathcal{E}]_{u_1} \otimes [\mathcal{E}]_{u_2}$ and $\mathcal{W}_{u_1,u_2}^* = [\mathcal{E}]_{u_1}^* \otimes [\mathcal{E}]_{u_2}^*$ for the dual space. For given functions $u_1, u_2 \in C_{+,1/2}$ satisfying $(U0)(U2)(U3)$, we have the following continuous inclusions:

$$\mathcal{W}_{u_1,u_2} \subset [\mathcal{E}_{p_1}]_{u_1} \otimes [\mathcal{E}_{p_2}]_{u_2} \subset L^2(\mathcal{E}_c', \mu_c) \subset [\mathcal{E}_{p_1}]_{u_1}^* \otimes [\mathcal{E}_{p_2}]_{u_2}^* \subset \mathcal{W}_{u_1,u_2}^*.$$

where $[\mathcal{E}_{p_1}]_{u_1}^* \otimes [\mathcal{E}_{p_2}]_{u_2}^*$ is the dual space of $[\mathcal{E}_{p_1}]_{u_1} \otimes [\mathcal{E}_{p_2}]_{u_2}$. In general, u_1 and u_2 are not necessarily the same functions. A Gel'fand triple $\mathcal{W}_{u_1,u_2} \subset L^2(\mathcal{E}_c', \mu_c) \subset \mathcal{W}_{u_1,u_2}^*$ is refered to as a *CKS-space with a weight sequence* $\alpha_{u_1}(l)\alpha_{u_2}(m)$. The bilinear form on $\mathcal{W}_{u_1,u_2}^* \times \mathcal{W}_{u_1,u_2}$ is denoted by $\langle\langle \cdot, \cdot \rangle\rangle_c$. Then

$$\langle\langle \Phi, \varphi \rangle\rangle_c = \sum_{n=0}^{\infty} l!m! \langle F_{l,m}, f_{l,m} \rangle,$$

and it holds that

$$|\langle\langle \Phi, \varphi \rangle\rangle_c| \leq \|\Phi\|_{-p_1,-p_2} \|\varphi\|_{p_1,p_2}$$

where

$$\|\Phi\|_{-p_1,-p_2}^2 = \sum_{l,m=0}^{\infty} \frac{1}{\ell_{u_1^*}(l)\ell_{u_2^*}(m)} |F_{l,m}|_{-p_1,-p_2}^2, \quad F_{l,m} \in (\mathcal{E}_c^{\otimes(l+m)})_{symm}^*. \tag{2.5}$$

2.4 Examples

The combinations of two functions out of following examples are applicable to our setting.

Example 2.1. Consider

$$u(r) = u^*(r) = e^r.$$

Then it is obvious to check that consditions (U0) (U2) (U3) are satisfied. This example produces to the *Hida-Kubo-Takenaka space* over the real Gaussian space. See [14, 21, 22, 26].

Example 2.2. For $0 \leq \beta < 1$, let u be the function defined by

$$u(r) = \exp\left[(1+\beta) r^{\frac{1}{1+\beta}}\right].$$

It is easy to check that u belongs to $C_{+,1/2}$ and satisfies conditions (U0) (U2) (U3). By Example 4.3 in [4], the dual Legendre transform u^* of u is given by

$$u^*(r) = \exp\left[(1-\beta)r^{\frac{1}{1-\beta}}\right].$$

This example is for the construction of the *Kodratiev-Streit space* over the real Gaussian space. See [16, 17, 23]

Example 2.3. Consider the function $v(r) = \exp\left[e^r - 1\right]$. Obviously, $v \in C_{+,1/2}$. Let $u = v^*$ be the dual Legendre transform of v. Then $u(0) = \sup_{s\geq 0} v(s)^{-1} = 1$ and it can be shown that u belongs to $C_{+,1/2}$ and is an increasing (log, x^2)-convex function on $[0,\infty)$ (See [7]). Hence $u \in C_{+,1/2}$ satisfies conditions (U1) and (U3). It is shown in Example 4.4 in [4] that u is equivalent to the function

$$w(r) = \exp\left[2\sqrt{r\log\sqrt{r}}\right].$$

("u is equivalent to v" means that there exist constants $a_1, a_2, b_1, b_2 > 0$ and $r_0 \in [0,\infty)$ such that $a_1 b_1^n u(r) \leq v(r) \leq a_2 b_2^n u(r)$ for all $r \geq r_0$.) Obviously, w satisfies condition (U2) and so u also satisfies condition (U2). On the other hand, we have the involution property $u^* = (v^*)^* = v$. This example can be applied to the Gel'fand triple $[\mathcal{E}]_u \subset (L^2) \subset [\mathcal{E}]_u^*$ for the following pair of functions:

$$u^*(r) = \exp\left[e^r - 1\right], \quad u(r) = (u^*)^*.$$

In general, we can consider the follwing general pair of functions:

$$\exp_k(r) = \exp(\exp(\cdots(\exp(r)))), \quad w_k(r) = \exp\left[2\sqrt{r\log_{k-1}\sqrt{r}}\right]$$

We refer the reader to papers [1, 2, 3, 4, 5, 7, 8, 9, 20].

3 Characterizations of Product measures and Pseudo-positive Operators

We shall define another norm as follows. Let $\mathcal{D}_{p_1,p_2}$ for $p_1, p_2 \geq 1$ be the space of all functions φ on $\mathcal{E}_c^* \times \mathcal{E}_c^*$ satisfying the following conditions:
(L1) φ is an analytic function on $\mathcal{E}_{p_1,c}^* \times \mathcal{E}_{p_2,c}^*$.
(L2) There exists a nonnegative constant C such that

$$|\varphi(x,y)|^2 \leq C u_1(|x|_{-p_1}^2) u_2(|y|_{-p_2}^2) \;\text{ for any } (x,y) \in \mathcal{E}_{p_1,c}^* \times \mathcal{E}_{p_2,c}^*.$$

For $\varphi \in \mathcal{D}_{p_1,p_2}$, its norm is defined by

$$\|\varphi\|_{p_1,p_2} := \sup_{(x,y)\in\mathcal{E}_{p_1,c}^*\times\mathcal{E}_{p_2,c}^*} |\varphi(x,y)| u_1(|x|_{-p_1}^2)^{-\frac{1}{2}} u_2(|y|_{-p_2}^2)^{-\frac{1}{2}}. \tag{3.1}$$

for a function $u \in C_{+,\log}$. Define the space $\mathcal{D}_{u_1,u_2}$ of test functions on $\mathcal{E}^* \times \mathcal{E}^*$ to be the projective limit of $\mathcal{D}_{p_1,p_2}$ as $p_1, p_2 \to \infty$. Let $\mathcal{D}^*_{u_1,u_2}$ be the dual space of $\mathcal{D}_{u_1,u_2}$.

Remark. This construction is motivated by Lee [25] and Asai et al. [5, 7, 8]. See also [15, 23] and references cited therein. Asai et al. [5, 7, 8] and Gannoun et al. [11] have considered the case of $u_2 \equiv 1$, independently. In addition, Ouerdiane studied similar situations and the case $u_1(r^2) = u_2(r^2) = \exp(k^{-1}r^k)$ where $1 \le k \le 2$.

Lemma 3.1. *If $u_1, u_2 \in C_{+,1/2}$ and an entire function $F(\xi,\eta)$ on $\mathcal{E}_c \times \mathcal{E}_c$ satisfies the growth condition*

$$|F(\xi,\eta)|^2 \le Cu_1^*(K_1|\xi|^2_{p_1})u_2^*(K_2|\eta|^2_{p_2}) \tag{3.2}$$

for a fixed positive $p_i \in \mathbf{R}$, then for $q_i > p_i$ with $K_i e^2 \|i_{q_i,p_i}\|^2_{HS} < 1$, there exists a kernel $\kappa_{l,m} \in (\mathcal{E}_c^{\otimes(l+m)})^*_{symm}$ such that

$$F(\xi,\eta) = \sum_{l,m=0}^{\infty} \langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle \tag{3.3}$$

and

$$|\kappa_{l,m}|^2_{-q_1,-q_2} \le C^2(K_1e^2\|i_{q_1,p_1}\|^2_{HS})^l(K_2e^2\|i_{q_2,p_2}\|^2_{HS})^m \ell_{u_1^*}(l)\ell_{u_2^*}(m). \tag{3.4}$$

Proof. Consider an entire function on $\mathbf{C}^{m+l}$

$$\psi = \psi(z_1, \cdots, z_m, w_1, \cdots, w_l) := F(z_1\xi_1 + \cdots + z_m\xi_m, w_1\eta_1 + \cdots + w_l\eta_l).$$

Define an $(l+m)$-linear functional $V_{l,m}$ on $\mathcal{E}_c \times \mathcal{E}_c$

$$V_{l,m}(\xi_1, \cdots, \xi_m, \eta_1, \cdots .\eta_l) := \frac{1}{l!m!} \frac{\partial^{l+m}\psi}{\partial z_1 \cdots \partial z_m \partial w_1 \cdots \partial w_l}\Bigg|_{\substack{z_1=\cdots=z_m=0\\ w_1=\cdots=w_l=0}}$$

$$= \frac{1}{l!m!}\frac{1}{(2\pi)^{l+m}} \prod_{j=1}^{m} \int_{|z_j|=r_j} \frac{dz_j}{z_j^2} \prod_{k=1}^{l} \int_{|w_k|=r_k} \frac{dw_j}{w_k^2} \psi(z_1, \cdots, z_m, w_1, \cdots, w_l).$$

Taking $r = r_1|\xi_1|_{p_1} = \cdots = r_n|\xi_m|_{p_1}$ and $s = s_1|\eta_1|_{p_2} = \cdots = s_l|\eta_l|_{p_2}$ we get

$$|V_{l,m}(\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_l)| \le C\frac{1}{l!m!} \frac{u_1^*(K_1m^2r^2)^{\frac{1}{2}}}{r^m} \frac{u_2^*(K_2l^2s^2)^{\frac{1}{2}}}{s^l} \times |\xi_1|_{p_1} \cdots |\xi_m|_{p_1} |\eta_1|_{p_2} \cdots |\eta_l|_{p_2}$$

by (3.2). Minimizing the right term, we have

$$|V_{l,m}(\xi_1, \cdots, \xi_m, \eta_1 \cdots, \eta_l)| \le CK_1^{\frac{m}{2}} K_2^{\frac{l}{2}} \frac{m^m}{m!}\frac{l^l}{l!} \ell_{u_1^*}(m)^{\frac{1}{2}} \ell_{u_2^*}(l)^{\frac{1}{2}} \times |\xi_1|_{p_1} \cdots |\xi_m|_{p_1} |\eta_1|_{p_2} \cdots |\eta_l|_{p_2}.$$

This shows that $V_{l,m}$ can be expressed in the form

$$V_{l,m}(\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_l) = \langle \kappa_{l,m}, \eta_1 \cdots \otimes \eta_l \otimes \zeta_1 \otimes \cdots \otimes \zeta_m \rangle$$

with $\kappa_{l,m} \in (\mathcal{E}_c^{\otimes(l+m)})^*_{symm}$,

$$|\kappa_{l,m}|^2_{-(p+q)} \le C^2(K_1e^2\|i_{q_1,p_1}\|^2_{HS})^l(K_2e^2\|i_{q_2,p_2}\|^2_{HS})^m \ell_{u_1^*}(l)\ell_{u_2^*}(m)$$

with finite Hilbert-Schmidt norm $K_ie^2\|i_{q_i,p_i}\|^2_{H.S.} < 1$ for any $q_i > p_i$. Therefore we derive

$$F(\xi, \eta) = \sum_{l,m=0}^{\infty} \langle \kappa_{l,m}, \eta^{\widehat{\otimes} l} \otimes \zeta^{\widehat{\otimes} m} \rangle.$$

□

Similarly, we have

Lemma 3.2. *If $u_1, u_2 \in C_{+,\log}$ and an entire function $F(\xi, \eta)$ on $\mathcal{E}_c' \times \mathcal{E}_c'$ satisfies the growth condition*

$$|F(\xi, \eta)|^2 \le Cu_1(K_1|\xi|^2_{-p_1})u_2(K_2|\eta|^2_{-p_2}) \tag{3.5}$$

for any $K_i, p_i \ge 0$ $(i = 1, 2)$ and some $C > 0$, then there exists a kernel $\kappa_{l,m} \in \mathcal{E}_c^{\widehat{\otimes}(l+m)}$ such that

$$F(\xi, \eta) = \sum_{l,m=0}^{\infty} \langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle \tag{3.6}$$

and

$$|\kappa_{l,m}|^2_{q_1,q_2} \le C^2(K_1e^2\|i_{p_1,q_1}\|^2_{HS})^l(K_2e^2\|i_{p_2,q_2}\|^2_{HS})^m \ell_{u_1}(l)\ell_{u_2}(m). \tag{3.7}$$

for any $q_i < p_i$ $(i = 1, 2)$ satisfying $K_ie^2\|i_{p_i,q_i}\|^2 < 1$.

Proof. We can prove this Lemma with modifications of the proof of the previous Lemma. Therefore, we omit the proof. □

Remark. Lemma 3.2 will be expected to characterize the continuous linear operator from $[\mathcal{E}]^*_{u_2}$ into $[\mathcal{E}]_{u_1}$ and expand it in terms of integral kernel operators [26, 27]. To our best knowledge, such consideratoins have not been done in any literature.

For $\Phi \in \mathcal{W}_{u_1,u_2}$, its *multiple S-transform* $S_m\Phi$ is defined to be the function

$$(S_m\Phi)(\xi, \eta) = \langle\!\langle \Phi, e^{\sqrt{2}\langle x,\xi\rangle - \frac{1}{2}\langle \xi,\xi\rangle} \otimes e^{\sqrt{2}\langle y,\eta\rangle - \frac{1}{2}\langle \eta,\eta\rangle} \rangle\!\rangle_c, \quad \xi, \eta \in \mathcal{E}_c. \tag{3.8}$$

For $\varphi \in L^2(\mathcal{E}_c', \mu')$, we have an integral representation of the multiple S-transform as

$$S_m\varphi(\xi, \eta) = \int_{\mathcal{E}^*}\int_{\mathcal{E}^*} \varphi(x + \sqrt{2}\xi, y + \sqrt{2}\eta)\mu'(dx)\mu'(dy). \tag{3.9}$$

Note that the multiple S-transform is nothing but a *symbol of operators* frequently used in white noise operator theory [26, 27]. Lemma 3.1, Equation (3.2) and S_m (Lemma 3.2, Equation (3.5) and S_m) give us the characterization theorem for $\mathcal{W}^*_{u_1,u_2}$ ($\mathcal{W}_{u_1,u_2}$), respectively, which generalize recent results in [8].

Proposition 3.3. *Suppose $u_1, u_2 \in C_{+,1/2}$ satisfy $(U0)(U2)(U3)$. Then the families of norms $\{|\!|\!|\cdot|\!|\!|_{p_1,p_2};\ p_1, p_2 \geq 0\}$ and $\{\|\cdot\|_{p_1,p_2};\ p_1, p_2 \geq 0\}$ are equivalent.*

Proof. First, we will show that for any $p_i \geq 1$, $i = 1, 2$, there exist $C \geq 0$ and $q_i > p_i$ such that $|\!|\!|\varphi|\!|\!|_{p_1,p_2} \leq C\|\varphi\|_{q_1,q_2}$. Let $p_i \geq 1$, $i = 1, 2$, be given. Since it has been proved [25] (see also [8, 23]) that every test function in $[\mathcal{E}]_u$ has an analytic extention, there exist $C \geq 0$ and $q_i \geq p_i$ such that for any $\varphi \in [\mathcal{E}_{q_1}]_{u_1} \otimes [\mathcal{E}_{q_2}]_{u_2}$

$$|\varphi(x, y)| \leq C\|\varphi\|_{q_1,q_2} u_1(|x|^2_{-p_1})^{\frac{1}{2}} u_2(|y|^2_{-p_2})^{\frac{1}{2}} \tag{3.10}$$

for any $(x, y) \in \mathcal{E}^*_{p_1,c} \times \mathcal{E}^*_{p_2,c}$. Hence it is derived by (3.1) and (3.10) that

$$\begin{aligned} |\!|\!|\varphi|\!|\!|_{p_1,p_2} &= \sup_{(x,y)\in\mathcal{E}^*_{p_1,c}\times\mathcal{E}^*_{p_2,c}} |\varphi(x, y)| u(|x|^2_{-p_1})^{-\frac{1}{2}} u(|y|^2_{-p_2})^{-\frac{1}{2}} \\ &\leq C\|\varphi\|_{q_1,q_2}. \end{aligned} \tag{3.11}$$

To prove the converse, The multiple S-transform of φ is given by

$$F(\xi, \eta) := S_m\varphi(\xi, \eta) = \int_{\mathcal{E}^*}\int_{\mathcal{E}^*} \varphi(x + \sqrt{2}\xi, y + \sqrt{2}\eta)\mu'(dx)\mu'(dy).$$

Then observe that for $q_i \geq 1$

$$\begin{aligned} |F(\xi, \eta)| &\leq \int_{\mathcal{E}^*}\int_{\mathcal{E}^*} |\varphi(x + \sqrt{2}\xi, y + \sqrt{2}\eta)|\mu'(dx)\mu'(dx) \\ &= \int_{\mathcal{E}^*}\int_{\mathcal{E}^*} |\varphi(x + \sqrt{2}\xi, y + \sqrt{2}\eta)| u\big(|x + \sqrt{2}\xi|^2_{-q_1}\big)^{-\frac{1}{2}} u\big(|y + \sqrt{2}\eta|^2_{-q_2}\big)^{-\frac{1}{2}} \\ &\qquad\qquad \times u\big(|x + \sqrt{2}\xi|^2_{-q_1}\big)^{\frac{1}{2}} u\big(|y + \sqrt{2}\eta|^2_{-q_2}\big)^{\frac{1}{2}} \mu'(dx)\mu'(dy) \\ &\leq |\!|\!|\varphi|\!|\!|_{q_1,q_2} \int_{\mathcal{E}^*}\int_{\mathcal{E}^*} u\big(|x + \sqrt{2}\xi|^2_{-q_1}\big)^{\frac{1}{2}} u\big(|y + \sqrt{2}\eta|^2_{-q_2}\big)^{\frac{1}{2}} \mu'(dx)\mu'(dy). \end{aligned}$$

By the condition $(U0)$, $u^{\frac{1}{2}}(r) \leq u(r)$ for all $r \geq 0$. Therefore,

$$|F(\xi, \eta)| \leq \int_{\mathcal{E}^*}\int_{\mathcal{E}^*} u\big(|x + \sqrt{2}\xi|^2_{-q_1}\big) u\big(|y + \sqrt{2}\eta|^2_{-q_2}\big)\mu'(dx)\mu'(dy)$$

By the condition $(U3)$, we have

$$\begin{aligned} u(|x + \sqrt{2}\xi|^2_{-q}) &\leq u\big((|x|_{-q} + |\sqrt{2}\xi|_{-q})^2\big) \\ &\leq u(4|x|^2_{-q})^{\frac{1}{2}} u(8|\xi|^2_{-q})^{\frac{1}{2}}. \end{aligned}$$

Thus, it is easy to get

$$|F(\xi,\eta)| \leq L \|\varphi\|_{q_1,q_2} u_1(8|\xi|^2_{-q_1})^{\frac{1}{2}} u_2(8|\eta|^2_{-q_2})^{\frac{1}{2}} \tag{3.12}$$

where

$$L = \int_{\mathcal{E}^*} \int_{\mathcal{E}^*} u(4|x|^2_{-q_1})^{\frac{1}{2}} u(4|y|^2_{-q_2})^{\frac{1}{2}} \mu'(dx)\mu'(dy) < \infty.$$

(Note that finiteness concerning L can be shown easily by simple estimation and the Fernique theorem [10, 23].) Then applying Lemma 3.2 with (3.12), we have

$$\|\varphi\|^2_{p_1,p_2} \leq L^2(1-8e^2\|i_{q_1,p_1}\|^2_{HS})^{-1}(1-8e^2\|i_{q_2,p_2}\|^2_{HS})^{-1}\|\varphi\|^2_{q_1,q_2}. \tag{3.13}$$

We complete the proof. □

Definition 3.4. A generalized function $\Phi \in \mathcal{W}^*_{u_1,u_2}$ is called *positive* if $\langle\langle \Phi, \varphi \rangle\rangle \geq 0$ for all nonnegative test functions $\varphi \in \mathcal{D}_{u_1,u_2}$.

Remark. Positivity of generalized functions in white noise context has been studied by Yokoi [31].

It is possible to give an alternative definition to Definition 3.4 as follows by the kernel theorem [30].

Definition 3.5. An operator $\Xi \in \mathcal{L}([\mathcal{E}]_{u_1}, [\mathcal{E}]^*_{u_2})$ is called *positive* in the sense of distributions if $\langle\langle \Xi\varphi_1, \varphi_2 \rangle\rangle_c \geq 0$ for all nonnegative test functions $\varphi_i \in [\mathcal{E}]_{u_i}$, $(i = 1,2)$. We call such an operator *pseudo-positive* operator

Notation. For locally convex spaces X, Y, let $\mathcal{L}(X,Y)$ denote the space of all continuous operators from X into Y equipped with the topology of uniform convergence on every bounded subset.

Theorem 3.6. *Suppose $u_1, u_2 \in C_{+,1/2}$ satisfy $(U0)(U2)(U3)$. A measure $\nu_1 \times \nu_2$ on $\mathcal{E}' \times \mathcal{E}'$ is a positive product Radon measure inducing a positive generalized function $\Phi_{\nu_1\times\nu_2} \in \mathcal{W}^*_{u_1,u_2}$ if and only if $\nu_1 \times \nu_2$ is supported in $\mathcal{E}^*_{p_1} \times \mathcal{E}^*_{p_2}$ for some $p_1, p_2 \geq 1$ and*

$$\int_{\mathcal{E}^*_{p_1}} \int_{\mathcal{E}^*_{p_2}} u_1(|x|^2_{-p_1})^{\frac{1}{2}} u_2(|y|^2_{-p_2})^{\frac{1}{2}} \nu_1(dx)\nu_2(dy) < \infty. \tag{3.14}$$

Proof. First we shall prove sufficiency. Suppose that $\nu_1 \times \nu_2$ is supported in $\mathcal{E}'_{p_1} \times \mathcal{E}'_{p_2}$ for some $p_1, p_2 \geq 0$ and Equation (3.14) holds. Then for any $\varphi(x,y) \in$

$\mathcal{W}_{u_1,u_2}$,

$$\begin{aligned}
&\int_{\mathcal{E}'_{p_1}}\int_{\mathcal{E}'_{p_2}} |\varphi(x,y)|\nu_1(dx)\nu_2(dy) \\
&= \int_{\mathcal{E}'_{p_1}}\int_{\mathcal{E}'_{p_2}} |\varphi(x,y)|u_1(|x|^2_{-p_1})^{-\frac{1}{2}}u_2(|y|^2_{-p_2})^{-\frac{1}{2}} \\
&\qquad\qquad\qquad \times u_1(|x|^2_{-p_1})^{\frac{1}{2}}u_2(|y|^2_{-p_2})^{\frac{1}{2}}\nu_1(dx)\nu_2(dy) \\
&\le \|\!|\varphi|\!\|_{p_1,p_2}\int_{\mathcal{E}'_{p_1}}\int_{\mathcal{E}'_{p_2}} u_1(|x|^2_{-p_1})^{\frac{1}{2}}u_2(|y|^2_{-p_2})^{\frac{1}{2}}\nu_1(dx)\nu_2(dy) \qquad (3.15)
\end{aligned}$$

With the help of Proposition 3.3, $\mathcal{W}_{u_1,u_2} \subset L^1(\mathcal{E}'_c, \nu_1\times\nu_2)$ and

$$\varphi \longmapsto \int_{\mathcal{E}'_{p_1}}\int_{\mathcal{E}'_{p_2}} \varphi(x,y)\nu_1(dx)\nu_2(dy) \qquad (3.16)$$

is a continuous linear functional on $\mathcal{W}_{u_1,u_2}$. Therefore, $\nu_1\times\nu_2$ is a positive product Radon measure which induces a positive generalized function $\Phi_{\nu_1\times\nu_2}$ in $\mathcal{W}^*_{u_1,u_2}$.

Conversely, suppose that $\nu_1\times\nu_2$ is a positive product Radon measure. Then for all $\varphi\in\mathcal{W}_{u_1,u_2}$,

$$\langle\!\langle \Phi_{\nu_1\times\nu_2}, \varphi\rangle\!\rangle = \int_{\mathcal{E}'}\int_{\mathcal{E}'} \varphi(x,y)\nu_1(dx)\nu_2(dy). \qquad (3.17)$$

is a continuous linear functional with respect to $\{\|\!|\cdot|\!\|^2_{q_1,q_2};\ p_1,p_2\ge 0\}$ by Proposition 3.3. Thus there exist constants $K, q_1, q_2 \ge 0$ such that for all $\varphi\in\mathcal{D}_{u_1,u_2}$

$$\left|\langle\!\langle \Phi_{\nu_1\times\nu_2}, \varphi\rangle\!\rangle\right| \le K\|\!|\varphi|\!\|_{q_1,q_2}. \qquad (3.18)$$

Let us define an analytic function ω on $\mathcal{E}'_{q_1c}\times\mathcal{E}'_{q_2,c}$ by

$$\omega(x,y) = \mathcal{L}_{u_1}(2^{-4}\langle x,x\rangle_{-q_1})\mathcal{L}_{u_2}(2^{-4}\langle y,y\rangle_{-q_2}),\ (x,y)\in\mathcal{E}'_{q_1c}\times\mathcal{E}'_{q_2,c} \qquad (3.19)$$

where $\langle\cdot,\cdot\rangle_{-q_i}$ is the bilinear pairing on $\mathcal{E}'_{q_i,c}$, $(i=1,2)$. On the other hand, Fact 2.1 (3) implies that

$$\begin{aligned}
|\omega(x,y)| &\le \mathcal{L}_{u_1}(2^{-4}|x|^2_{-q_1})\mathcal{L}_{u_2}(2^{-4}|x|^2_{-q_2}) \\
&\le \frac{2e}{\log 2}u_1(|x|^2_{-q_1})^{\frac{1}{2}}u_2(|x|^2_{-q_2})^{\frac{1}{2}}. \qquad (3.20)
\end{aligned}$$

(It is easy to find an increasing function v equivalent to a function u.) This implies that $\omega\in\mathcal{D}_{p_1,p_2}$. Thus, from Equation (3.18) with $\varphi=\omega$ we obtain that

$$\left|\int_{\mathcal{E}'}\omega(x,y)\nu_1(dx)\nu_2(dy)\right| \le K\frac{2e}{\log 2}. \qquad (3.21)$$

Due to Equation (3.21), we have

$$\int_{\mathcal{E}'} \omega(x,y)\nu_1(dx)\nu_2(dy) < \infty.$$

However Fact 2.1 (2) says that $u_1(r_1)u_2(r_2) \le Cw(4r_1, 4r_2)$. Therefore,

$$\int_{\mathcal{E}'} u_1(2^{-6}|x|^2_{-q_1})u_2(2^{-6}|y|^2_{-q_2})\nu_1(dx)\nu_2(dy) < \infty.$$

By choosing an appropriate $p_i > q_i$ $(i = 1, 2)$ satisfying $\rho^{2(p_i - q_2)} \le 2^{-6}$, so that $|\cdot|^2_{-p_i} \le 2^{-6}|\cdot|^2_{-q_i}$ $(i = 1, 2)$. Therefore we get the assertion. $\square$

Theorem 3.7. *Suppose $u_1, u_2 \in C_{+,1/2}$ satisfy $(U0)(U2)(U3)$. A measure $\nu_1 \times \nu_2$ on $\mathcal{E}' \times \mathcal{E}'$ is a positive product Radon measure inducing a pseudo-positive operator $\Xi \in \mathcal{L}([\mathcal{E}]_{u_1}, [\mathcal{E}]^*_{u_2})$ if and only if $\nu_1 \times \nu_2$ is supported in $\mathcal{E}^*_{p_1} \times \mathcal{E}^*_{p_2}$ for some $p_1, p_2 \ge 1$ and*

$$\int_{\mathcal{E}^*_{p_1}} \int_{\mathcal{E}^*_{p_2}} u_1(|x|^2_{-p_1})^{\frac{1}{2}} u_2(|y|^2_{-p_2})^{\frac{1}{2}} \nu_1(dx)\nu_2(dy) < \infty. \tag{3.22}$$

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